Elliptic curves and modularity

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Perfect squares and quadratic residues

Question

What are the last digits of perfect squares?

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Harder question

Let $\ell \geq 5$ be a prime number. Can 3 be the last digit of a perfect square in base ℓ? When does the polynomial

$$x^2 - 3$$

split into two distinct linear factors modulo ℓ?

The law of quadratic reciprocity

Let p and ℓ be distinct odd primes. The law of quadratic reciprocity relates whether

- p is a quadratic residue modulo ℓ

to whether

- ℓ is a quadratic residue modulo p.

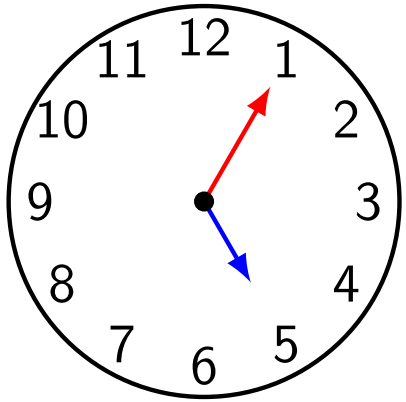
$$\left(\frac{p}{\ell}\right) \cdot \left(\frac{\ell}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{\ell-1}{2}}$$

It was conjectured by Euler and Legendre and first proved by Gauss in 1796.

The law of quadratic reciprocity

Consequence: whether 3 can be the last digit of a perfect square in base ℓ only depends on ℓ modulo $3 \cdot 4 = 12$.

- For ℓ equal to 13, 37, 61 and 1093, the answer is "Yes".
- For ℓ equal to 17, 29, 41 and 1637, the answer is "No".



What is the most general reciprocity law?

Higher-dimensional reciprocity

The infinite product

$$f(q) := q \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{11n})^2$$

$$= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 + \dots$$

and the Diophantine equation

$$E : y^2 + y = x^3 - x^2$$

seem to know about each other in a mysterious way.

Higher-dimensional reciprocity

- The coefficient a_ℓ of q^ℓ in the expansion of $f(q)$ takes the values:

ℓ	2	3	5	7	13	17	19	23	29
a_ℓ	-2	-1	1	-2	4	-2	0	-1	0

- The number N_ℓ of solutions to $y^2 + y \equiv x^3 - x^2 \pmod{\ell}$ takes the values:

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- We always seem to have $a_\ell = \ell - N_\ell$.

Higher-dimensional reciprocity laws

- The power series $f(q)$ is the Fourier expansion of a modular form.
- The Diophantine equation E represents an elliptic curve over $\mathbb{Q}$.

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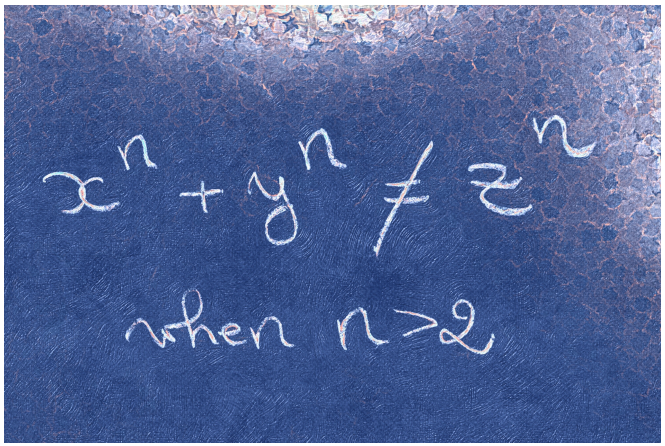


Elliptic curves

An elliptic curve $E/\mathbb{Q}$ is a smooth projective curve of genus 1 together with a specified rational point.

Its torsion points $E[p^n] \simeq (\mathbb{Z}/p^n\mathbb{Z})^2$ have an action of the absolute Galois group $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

As n varies, these assemble into a Galois representation

$$\rho_E : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Q}_p).$$

More generally, the étale cohomology of algebraic varieties defined over $\mathbb{Q}$ is a source of representations of $G_{\mathbb{Q}}$.

Modular forms

A modular form is a holomorphic function on the upper-half plane

$$\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}\ z > 0\}$$

that satisfies many symmetries and a growth condition.

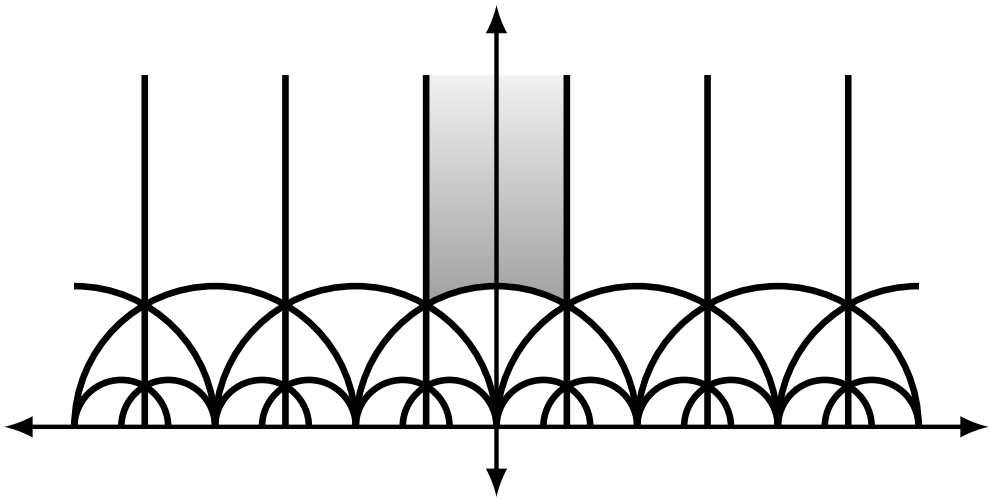
There is a transitive action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}^2$:

$$z \mapsto \frac{az+b}{cz+d} \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

This gives the identification of $\mathbb{H}^2$ with the symmetric space for the group SL_2.

Modular curves

Modular curves are quotients $\Gamma\backslash\mathbb{H}^2$, where $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup.



The quotients $\Gamma\backslash\mathbb{H}^2$ arise from algebraic curves defined over number fields. Their étale cohomology is a source of Galois representations

$$f \mapsto \rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Q}_p).$$

Modularity

We say that an elliptic curve $E/\mathbb{Q}$ is *modular* if there exists a modular form f together with an isomorphism of Galois representations

$$\rho_f \simeq \rho_E : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Q}_p)$$

This implies the explicit reciprocity law $a_\ell = \ell - N_\ell$ by taking traces at distinguished conjugacy classes $\{\mathrm{Frob}_\ell\} \in G_{\mathbb{Q}}$.

Theorem 1 (Wiles, Taylor–Wiles 1995, Breuil–Conrad–Diamond–Taylor 2001)

Every elliptic curve $E/\mathbb{Q}$ is modular.

Modularity

Theorem 2 (Freitas–Le Hung–Siksek 2013)

Let F be a real quadratic field. Every elliptic curve E/F is modular.

This relies on improvements to the Taylor–Wiles method, particularly due to Kisin, and on a sophisticated analysis of quadratic points on modular curves.

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Theorem 3 (C–Newton 2023)

Every elliptic curve $E/\mathbb{Q}(i)$ is modular.

This relies on work of Calegari–Geraghty, Scholze, Allen–Khare–Thorne and many others.

Arithmetic hyperbolic 3-manifolds

The symmetric space for $\mathrm{SL}_2/\mathbb{Q}(i)$ is hyperbolic 3-space.

If $\Gamma \subset \mathrm{SL}_2(\mathbb{Z}[i])$ is a sufficiently small congruence subrgroup, the $\Gamma\backslash\mathbb{H}^3$ are arithmetic hyperbolic 3-manifolds. They do not have the structure of algebraic varieties!

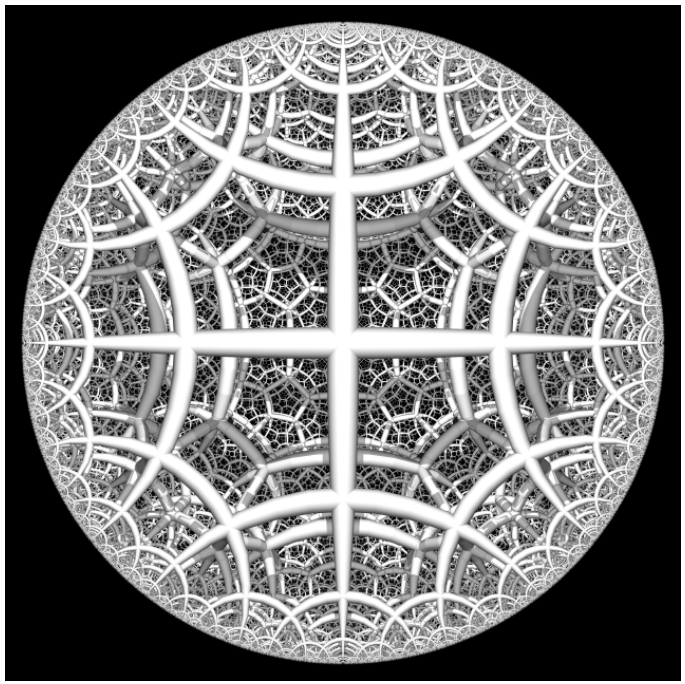

Image credit: http://kim.oyhus.no/icosians.html

The Langlands program

The modularity of elliptic curves is an instance of the global Langlands correspondence. This matches:

$$\{\text{spectral data}\} \text{ – seen on the automorphic side}$$

with

$$\{\text{arithmetic data}\} \text{ – seen on the Galois side.}$$

There are many instances of Langlands correspondences, including some with a more geometric flavour. It is an exciting and rapidly evolving field!

Thank you!

Career path and recognition

- PhD Harvard University (2007–2012).
- Postdocs: University of Chicago (2012–2013), Princeton / IAS (2013–2016), University of Bonn (2016–2017).
- Permanent positions: Imperial (2017–), University of Bonn (2022–).
- Professional recognition: LMS Whitehead Prize (2018), ERC Starting Grant (2019), EMS Prize (2020), New Horizons Prize (2023).

Things people have said to / about me

- That sounds complicated! You should leave it to the experts. (Professor during my PhD)
- Most of her papers are collaborative. The only single-author papers are from her PhD and they demonstrate technical strength but are not very creative. I am skeptical that she will become a leader in the field. (ERC reviewer)
- I guess, for now, you are a better mathematician than me. (Postdoc hired on my ERC grant)

Some advice

- Be aware of your strengths and weaknesses.
- Think about the long-term and aim for a growth mindset.
- Find (or build!) your community.